

# Foundations Note 11

## Continuity

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### 11.1 Limits of Functions

**Note:** In this note the symbol  $\Rightarrow$  is used to denote the sentential connective  $\rightarrow$  and also to denote a tautological implication (logical consequence). The symbol  $\Leftrightarrow$  will be used analogously (for esthetic symmetry). The purpose is to release  $\rightarrow$  from performing triple duty. Thus,  $\rightarrow$  will be used to denote limits (e.g.:  $1/x \rightarrow 0$  as  $x \rightarrow +\infty$ ) and as a function symbol (e.g.:  $f : A \rightarrow B$ ).

**11.1.1 Definition.** Let  $S$  be a subset of  $\mathbf{R}$  and let  $c$  be a limit point of  $S$ . Given a function  $f : S \rightarrow \mathbf{R}$ , we say that the *limit*  $\lim_{x \rightarrow c} f(x)$  *exists* if there is a number  $L$  such that given any  $\epsilon > 0$  there is some  $\delta > 0$  such that

$$(1) \quad (\forall x \in S)(0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon).$$

If the limit exists we write  $\lim_{x \rightarrow c} f(x) = L$ , or  $f(x) \rightarrow L$  as  $x \rightarrow c$  (“ $f(x)$  tends to  $L$  as  $x$  tends to  $c$ ,” or “ $f(x)$  approaches  $L$  as  $x$  approaches  $c$ ”).

If the limit exists for all  $c \in S$  and for a given  $\epsilon > 0$  the same  $\delta > 0$  can be used for all  $c \in S$ , we say that  $\lim_{x \rightarrow c} f(x)$  exists *uniformly on*  $S$ .

**11.1.2 Remark.** If  $S$  is an open subset of  $\mathbf{R}$  we can replace 1.1(1) with

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

In other words, if  $\delta$  in 1.1(1) is so large that there is an  $x$  not in  $S$  such that  $0 < |x - c| < \delta$ , specify a smaller  $\delta$ . Since  $S$  is open, there is an open interval  $(a, b)$  such that  $c \in (a, b) \subset S$ . Since  $a < c < b$ , choose  $0 < \delta \leq \min\{c - a, b - c\}$ . Then,

$$|x - c| < \delta \Rightarrow x \in S.$$

Note that the omission of the quantifier  $(\forall x \in S)$  simply means that  $x \in \mathbf{R}$ .

**11.1.3 Remarks.** Keep in mind that it does not make any sense to speak of a limit existing uniformly at a point. It is very important to understand the difference between  $\lim_{x \rightarrow c} f(x)$  just existing for all  $c \in S$  and existing uniformly on  $S$ . To say that  $\lim_{x \rightarrow c} f(x)$  exists for all  $c \in S$  means

$$(1) \quad (\forall c \in S)(\exists L)(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in S)(0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon).$$

Here,  $\delta$  may depend on  $c$  and on  $\epsilon$ . To say that  $\lim_{x \rightarrow c} f(x)$  exists uniformly on  $S$  means

$$(2) \quad (\forall \epsilon > 0)(\exists \delta > 0)(\forall c \in S)(\exists L)(\forall x \in S)(0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon).$$

Now,  $\delta$  may depend on  $\epsilon$ , but the same  $\delta$  must work for all  $c$ .

**11.1.4 Examples.**

1. Note that  $f$  need not be defined at the point  $c$ . For example, if

$$f : (-\infty, 0) \cup (0, +\infty) \rightarrow \mathbf{R} : x \mapsto 1,$$

we have  $\lim_{x \rightarrow 0} f(x) = 1$ ; but  $f(0)$  is not defined.

2. For  $f : (-\infty, 0) \cup (0, +\infty) \rightarrow \mathbf{R} : x \mapsto 1/x$ ,  $\lim_{x \rightarrow 0} f(x)$  does not exist.
3. If  $f$  is defined at  $c$ , the limit may exist, but need not equal  $f(c)$ . For example, consider  $f : \mathbf{R} \rightarrow \mathbf{R}$  given by

$$f(x) = \begin{cases} 1, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then,  $\lim_{x \rightarrow 0} f(x) = 1$ , but  $f(0) = 0$ .

4. The limit may fail to exist even if the function is bounded and  $f(c)$  is defined. Consider  $f : \mathbf{R} \rightarrow \mathbf{R}$  given by

$$f(x) = \begin{cases} \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

then  $\lim_{x \rightarrow 0} f(x)$  does not exist.

5. The limit may nor exist anywhere. If  $f : \mathbf{R} \rightarrow \mathbf{R}$  is given by

$$f(x) = \begin{cases} 1, & x \text{ rational}, \\ 0, & x \text{ irrational}, \end{cases}$$

then  $\lim_{x \rightarrow c} f(x)$  does not exist for any  $c$ .

**11.1.5 Example.** Consider  $f : [0, +\infty) \rightarrow \mathbf{R} : x \mapsto \sqrt{x}$ . Then  $\lim_{x \rightarrow c} f(x)$  exists for all  $c \geq 0$ . To prove this statement we use a so-called  $\epsilon$ - $\delta$  argument, which goes as follows.

Let  $\epsilon > 0$  be given. First, suppose that  $c \neq 0$ . Choose  $\delta = \epsilon\sqrt{c}$ . For  $x \geq 0$  we have

$$\begin{aligned} 0 < |x - c| < \delta &\Rightarrow |\sqrt{x} - \sqrt{c}| = \left| (\sqrt{x} - \sqrt{c}) \cdot \frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}} \right| \\ &= \frac{|x - c|}{\sqrt{x} + \sqrt{c}} \\ &\leq \frac{|x - c|}{\sqrt{c}} \\ &< \frac{\delta}{\sqrt{c}} \\ &= \epsilon. \end{aligned}$$

If  $c = 0$ , choose  $\delta = \epsilon^2$ . For  $x \geq 0$  we have

$$0 < |x - 0| < \delta \Rightarrow |\sqrt{x} - 0| < \sqrt{\delta} = \epsilon.$$

**11.1.6 Remark.** Did we prove that the limit exists uniformly on  $[0, +\infty)$ ? No, we did not (but it does). It is not entirely clear how to choose  $\delta$  independently of  $c$ . This situation shall be remedied beautifully and simply soon. In any case,  $\epsilon$ - $\delta$  arguments can get to be a bit ugly. The next one is worse than the previous one.

**11.1.7 Example.** Consider  $f : (0, +\infty) \rightarrow \mathbf{R} : x \mapsto 1/x$ . The limit  $\lim_{x \rightarrow c} f(x)$  exists for all  $c > 0$ . The  $\epsilon$ - $\delta$  argument to prove this statement goes as follows.

Given  $c > 0$  and  $\epsilon > 0$ , choose  $\delta = \min\{\epsilon c^2/2, c/2\}$ . Then,  $\delta \leq c/2$  and  $\delta \leq \epsilon c^2/2$ . Further, we note that

$$|x - c| < \delta \Rightarrow -\delta < x - c \Rightarrow c - \delta < x.$$

For  $x > 0$  we have

$$0 < |x - c| < \delta \Rightarrow \left| \frac{1}{x} - \frac{1}{c} \right| = \frac{|x - c|}{xc} < \frac{|x - c|}{(c - \delta)c} \leq \frac{\delta}{(c - c/2)c} = \frac{\epsilon c^2/2}{c^2/2} = \epsilon.$$

**11.1.8 Example.** The domain of the function in the above definition need not be open or perfect. Consider  $f : \mathbf{N} \rightarrow \mathbf{R} : x \mapsto 2x$ . Since  $\mathbf{N}$  has no limit points,  $\lim_{x \rightarrow c} f(x)$  does not exist at any point of  $\mathbf{N}$ . On the other hand, for  $g : \{1, 1/2, 1/3, \dots\} \rightarrow \mathbf{R} : x \mapsto 2x$  we have  $\lim_{x \rightarrow 0} g(x) = 0$ .

**11.1.9 Remark.** How does one prove that a limit does not exist? To assert that  $\lim_{x \rightarrow c} f(x)$  exists is to assert

$$(1) \quad (\exists L)(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in S)(0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon).$$

Recall that if  $p$  and  $q$  are propositions, the negation  $\neg(p \Rightarrow q)$  is equivalent to  $p \wedge \neg q$ . Therefore, the negation

$$\neg(0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon)$$

is equivalent to the statement

$$0 < |x - c| < \delta \wedge |f(x) - L| \geq \epsilon.$$

Accordingly, the negation of (1), i.e.: “ $\lim_{x \rightarrow c} f(x)$  does not exist,” is

$$(2) \quad (\forall L)(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in S)(0 < |x - c| < \delta \wedge |f(x) - L| \geq \epsilon).$$

We shall put this to use below.

**11.1.10 Example.** We show that if  $f : (0, +\infty) \rightarrow \mathbf{R} : x \mapsto 1/x$ , then  $\lim_{x \rightarrow 0} f(x)$  does not exist. We need to show that for arbitrary  $L$  there is some  $\epsilon > 0$  such that given any  $\delta > 0$  we can choose  $x > 0$  such that

$$0 < |x - 0| < \delta \wedge \left| \frac{1}{x} - L \right| \geq \epsilon.$$

Surely, we can choose  $x > 0$  so small that  $1/x > L + \epsilon$  and such that  $x < \delta$ . With this  $x$  we have

$$\left| \frac{1}{x} - L \right| > \epsilon.$$

**11.1.11 Exercises.**

1. Consider  $f : (0, +\infty) \rightarrow \mathbf{R} : x \mapsto 1/x^2$ . Prove that  $\lim_{x \rightarrow c} f(x)$  exists for all  $c > 0$  and does not exist for  $c = 0$ .
2. Consider  $f : \mathbf{R} \rightarrow \mathbf{R}$  given by

$$f(x) = \begin{cases} x, & x \text{ rational,} \\ 0, & x \text{ irrational.} \end{cases}$$

Prove that  $\lim_{x \rightarrow c} f(x)$  exists for  $c = 0$  and does not exist for  $c \neq 0$ .

**11.1.12 Theorem.** Suppose  $S$  is a subset of  $\mathbf{R}$ ,  $c$  is a limit point of  $S$ , and  $f : S \rightarrow \mathbf{R}$ . Then

$$(1) \quad \lim_{x \rightarrow c} f(x) = L$$

if and only if for every sequence  $\{x_n\}$  in  $S$  converging to  $c$  and such that  $x_n \neq c$  we have

$$(2) \quad \lim_{n \rightarrow \infty} f(x_n) = L.$$

*Proof.* **We prove that (1) implies (2)** Let  $f : S \rightarrow \mathbf{R}$  be given. Suppose (1) holds for some  $L$  and some limit point  $c$  of  $S$ . For every  $\epsilon > 0$ , there is a  $\delta > 0$  such that for all  $x \in S$

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

With the above  $\delta$ , for any sequence  $\{x_n\}$  in  $S$  converging to  $c$  and such that  $x_n \neq c$  we have some  $N$  such that

$$n > N \Rightarrow |x_n - c| < \delta.$$

Since  $x_n \neq c$ , we have  $|x_n - c| < \delta \Rightarrow 0 < |x_n - c| < \delta$ . Further, since  $x_n \in S$  also, we have

$$n > N \Rightarrow |x_n - c| < \delta \Rightarrow 0 < |x_n - c| < \delta \Rightarrow |f(x_n) - L| < \epsilon,$$

showing that  $\lim_{n \rightarrow \infty} f(x_n) = L$ .

**We prove that (2) implies (1)** We do so by contraposition. We show that if (1) is false then (2) is false. Suppose, therefore, that (1) is false for some limit point  $c$  of  $S$ . Then,

$$(\forall L)(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in S)(0 < |x - c| < \delta \wedge |f(x) - L| \geq \epsilon).$$

Choose any  $L$ . To this  $L$  there corresponds some  $\epsilon > 0$  such that, for  $n = 0, 1, 2, \dots$ , we have

$$(\exists x_n \in S) \left( 0 < |x_n - c| < \frac{1}{n+1} \wedge |f(x_n) - L| \geq \epsilon \right).$$

This defines a sequence  $\{x_n\}$  of limit points of  $S$  with  $x_n \neq c$ . To see that it converges to  $c$ , let  $\eta > 0$  be given. With  $N$  such that  $1/N \geq \eta$ , we have

$$n > N \Rightarrow |x_n - c| < \frac{1}{n+1} < \frac{1}{N} = \eta.$$

Since we also have  $|f(x_n) - L| \geq \epsilon$ , (2) is false for our choice of  $L$ . Since  $L$  is arbitrary, (2) is false for all  $L$ .  $\square$

**11.1.13 Corollary.** If  $\lim_{x \rightarrow c} f(x)$  exists, it is unique.

*Proof.* Sequences have unique limits [Note 10, Proposition 2.4], and, in view of the equivalence proved in the theorem, functions must have unique limits also.  $\square$

### 11.1.1 Properties of Limits

**11.1.14 Definition.** Let  $S$  be a subset of  $\mathbf{R}$ , let  $\alpha$  be any real number and let  $f, g : S \rightarrow \mathbf{R}$ . We define the functions  $\alpha + f, \alpha f, f + g, fg, |f|, -f : S \rightarrow \mathbf{R}$  and  $f/g : \{x \in S \mid g(x) \neq 0\} \rightarrow \mathbf{R}$  by

$$(1) \quad (\alpha + f)(x) = \alpha + f(x);$$

$$(2) \quad (\alpha f)(x) = \alpha f(x);$$

$$(3) \quad (f + g)(x) = f(x) + g(x);$$

$$(4) \quad (fg)(x) = f(x)g(x);$$

$$(5) \quad (f/g)(x) = f(x)/g(x);$$

$$(6) \quad |f|(x) = |f(x)|;$$

$$(7) \quad (-f)(x) = -f(x).$$

**11.1.15 Theorem.** Let  $S$  be a subset of  $\mathbf{R}$  and let  $c$  be a limit point of  $S$ . Let  $f, g : S \rightarrow \mathbf{R}$ . Suppose

$$\lim_{x \rightarrow c} f(x) = L_1 \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = L_2.$$

Then, for any real number  $\alpha$ ,

- (1)  $\lim_{x \rightarrow c}(\alpha + f)(x) = \alpha + L_1$ ;
- (2)  $\lim_{x \rightarrow c}(\alpha f)(x) = \alpha L_1$ ;
- (3)  $\lim_{x \rightarrow c}(f + g)(x) = L_1 + L_2$ ;
- (4)  $\lim_{x \rightarrow c}(fg)(x) = L_1 L_2$ ;
- (5)  $\lim_{x \rightarrow c}(f/g)(x) = L_1/L_2$ , provided  $L_2 \neq 0$ ;
- (6)  $\lim_{x \rightarrow c}|f|(x) = |L_1|$ ;
- (7)  $\lim_{x \rightarrow c}(-f)(x) = -L_1$ .

*Proof.* The proof is an immediate consequence of Theorem 5.1 in *Note 10* and Theorem 1.12. □

## 11.2 Continuous Functions

**11.2.1 Definition.** Let  $S$  be a subset of  $\mathbf{R}$ . The function  $f : S \rightarrow \mathbf{R}$  is said to be *continuous at*  $c \in S$  if given any  $\epsilon > 0$  there is a  $\delta > 0$  such that

$$x \in S \wedge |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon.$$

Note that  $c$  need not be a limit point of  $S$ . If  $f$  is not continuous at  $c$ , we say that  $f$  is *discontinuous at*  $c$ .

If  $f$  is continuous at every point  $c \in S$ , we say that  $f$  is *continuous on*  $S$  (or just that it is *continuous*). If given any  $\epsilon > 0$  there is a  $\delta > 0$  such that

$$(\forall c \in S)(x \in S \wedge |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon),$$

we say that  $f$  is *uniformly continuous on*  $S$  (or just that it is *uniformly continuous*).

**11.2.2 Remark.** In the above definition, if  $c$  is a limit point of  $S$ , then continuity at  $c$  is equivalent to  $\lim_{x \rightarrow c} f(x) = f(c)$ . If  $c$  is an isolated point of  $S$ , then  $f$  is always continuous at  $c$ . Indeed, if  $c$  is an isolated point, then, for  $\delta > 0$  small enough, the only  $x \in S$  that satisfies  $|x - c| < \delta$  can only be the point  $c$ .

**11.2.3 Remark.** If  $S$  is perfect, the difference between continuity on  $S$  and uniform continuity on  $S$  is the same as the difference between  $\lim_{x \rightarrow c} f(x)$  simply existing for every  $c \in S$  and existing uniformly on  $S$ .

In general, the function  $f : S \rightarrow \mathbf{R}$  is continuous on  $S$  if

$$(\forall c \in S)(\forall \epsilon > 0)(\exists \delta > 0)(x \in S \wedge |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon).$$

Note that  $\delta$  may depend on  $c$  and on  $\epsilon$ . To say that  $\lim_{x \rightarrow c} f(x)$  exists uniformly on  $S$  means

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall c \in S)(x \in S \wedge |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon).$$

Note that  $\delta$  may depend on  $\epsilon$ , but the same  $\delta$  must work for all  $c$ . Clearly, uniform continuity implies continuity.

**11.2.4 Remark.** Consider the function  $f : [0, +\infty) \rightarrow \mathbf{R} : x \mapsto \sqrt{x}$ . In Example 1.5 we proved that  $\lim_{x \rightarrow c} f(x)$  exists for every  $c \geq 0$ . In other words, since  $[0, +\infty)$  is perfect, we proved that  $f$  is continuous on  $[0, +\infty)$ . Does it follow, for instance, that if a nonnegative sequence  $\{x_n\}$  converges to  $L$  the sequence  $\{\sqrt{x_n}\}$  converges to  $\sqrt{L}$ ? Yes it does, as shown below.

**11.2.5 Theorem.** Let  $S$  be a subset of  $\mathbf{R}$ . Let  $\{x_n\}$  be a sequence in  $S$  that converges to  $L$  in  $S$ . Suppose that  $f : S \rightarrow \mathbf{R}$  is continuous at  $L$ . Then  $\{f(x_n)\}$  converges to  $f(L)$ .

*Proof.* Let  $\{x_n\}$  and  $f$  be given. Since  $f$  is continuous at  $L$ , given any  $\epsilon > 0$  there is a  $\delta > 0$  such that for any  $x \in S$  we have

$$|x - L| < \delta \Rightarrow |f(x) - f(L)| < \epsilon.$$

Since  $\{x_n\}$  converges to  $L$ , there is some  $N$  such that

$$n > N \Rightarrow |x_n - L| < \delta.$$

Therefore,

$$n > N \Rightarrow |x_n - L| < \delta \Rightarrow |f(x_n) - f(L)| < \epsilon,$$

showing that  $\{f(x_n)\}$  converges to  $f(L)$ . □

**11.2.6 Remark.** The above theorem states that, with the right hypotheses, if  $\{x_n\}$  converges to  $L$ , then  $\{f(x_n)\}$  converges to  $f(L)$ . That is,

$$\lim_{n \rightarrow \infty} f(x_n) = f(L) = f\left(\lim_{n \rightarrow \infty} x_n\right).$$

In this sense, continuity is a property that can be stated informally as “the limit of the  $f$ ’s is the  $f$  of the limit.”

**11.2.7 Example.** Let us agree to write every rational number  $x$  in the form  $x = m/n$ , with  $n > 0$  and where  $m$  and  $n$  are integers without common divisors. Further, we agree to take  $n = 1$  if  $x = 0$ . Define the function  $f : \mathbf{R} \rightarrow \mathbf{R}$  by

$$f(x) = \begin{cases} 1/n, & x \text{ rational,} \\ 0, & x \text{ irrational,} \end{cases}$$

where  $x = m/n$ , as stipulated above. We prove that  $f$  is continuous at every irrational and discontinuous at every rational.

Suppose  $c$  is rational. then,  $f(c) > 0$ . Let  $\{x_k\}$  denote a sequence of irrationals that converges to  $c$ . We have  $f(x_k) = 0$  for all  $k$ , giving

$$\lim_{k \rightarrow \infty} f(x_k) = 0 \neq f(c).$$

Hence,  $f$  is not continuous at  $c$ .

Suppose that  $c$  is irrational. For any  $\epsilon > 0$  there is some positive integer  $j$  such that  $1/j < \epsilon$ . In the interval  $(c - 1, c + 1)$  there are finitely many rationals  $m/n$  of the form stipulated above with  $n < j$ . To see the truth of this statement, consider all rationals  $m/n$  of the form stipulated such that

$$c - 1 < \frac{m}{n} < c + 1,$$

with  $0 < n < j$ . Equivalently, we must have

$$n(c - 1) < m < n(c + 1) \wedge 0 < n < j.$$

Since  $j$  and  $c$  are fixed, there are only finitely many integers  $m$  and  $n$  that satisfy the above.

Therefore, for all sufficiently small  $\delta > 0$  the interval  $(c - \delta, c + \delta)$  contains no rationals  $m/n$  with  $n < j$ . In other words, for  $|x - c| < \delta$  we have  $|f(x)| = 0$  if  $x$  is irrational, or, if  $x = m/n$  we have  $|f(x)| = 1/n$  with  $n \geq j$ . Hence,

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| = |f(x)| \leq \frac{1}{j} < \epsilon,$$

showing that  $f$  is continuous at  $c$ .

## 11.2.1 Hölder Continuity

**11.2.8 Definition.** Let  $I$  denote a real interval. The function  $f : I \rightarrow \mathbf{R}$  is called *Hölder continuous with exponent  $\alpha$  and constant  $\gamma$*  if there are constants  $\alpha$  and  $\gamma$ ,  $0 < \alpha \leq 1$ , such that

$$|f(x) - f(y)| \leq \gamma|x - y|^\alpha$$

for all  $x$  and  $y$  in  $I$ .

**11.2.9 Proposition.** Let  $0 < \alpha \leq 1$ . The function  $f : [0, +\infty) \rightarrow \mathbf{R} : x \mapsto x^\alpha$  is Hölder continuous.

*Proof.* Note that  $0 < \alpha \leq 1$  and  $0 \leq r \leq 1$  imply  $0 \leq r \leq r^\alpha \leq 1$ , which implies  $0 \leq 1 - r^\alpha \leq 1 - r \leq 1$ . Let  $x$  and  $y$  be such that  $0 < y \leq x$ , so that with  $r = y/x$  we have  $0 \leq r \leq 1$ . Then, we have

$$0 \leq x^\alpha - y^\alpha = x^\alpha \left[ 1 - \left( \frac{y}{x} \right)^\alpha \right] \leq x^\alpha \left( 1 - \frac{y}{x} \right) \leq x^\alpha \left( 1 - \frac{y}{x} \right)^\alpha = (x - y)^\alpha.$$

Now consider arbitrary nonnegative  $x$  and  $y$ . If either of them is zero, there is nothing to prove. Otherwise, we may suppose that  $x \geq y$  (if not, interchange  $x$  and  $y$ ). Then, from the above,  $|x^\alpha - y^\alpha| \leq |x - y|^\alpha$ .  $\square$

**11.2.10 Proposition.** A Hölder continuous function is uniformly continuous.

*Proof.* Suppose that  $I$  is a real interval and  $f : I \rightarrow \mathbf{R}$  is Hölder continuous with exponent  $\alpha$  and constant  $\gamma$ . Given  $\epsilon > 0$ , choose  $\delta = \epsilon^{1/\alpha}/\gamma$ . Then, for any  $c \in I$ , we have

$$(x \in I) \wedge (|x - c| < \delta) \Rightarrow |f(x) - f(c)| \leq \gamma|x - c|^\alpha < (\epsilon^{1/\alpha})^\alpha = \epsilon.$$

$\square$

**11.2.11 Remark.** From the last two propositions it follows that

$$f : [0, +\infty) \rightarrow \mathbf{R} : x \mapsto \sqrt{x}$$

is uniformly continuous.

**11.2.12 Remark.** In view of the above result, we can say more about real sequences. Not only do we have that if  $\{x_n\}$  is a sequence in  $[0, +\infty)$  that converges to  $L$  then  $\{\sqrt{x_n}\}$  is a sequence that converges to  $\sqrt{L}$ ; but also that  $\{x_n^\alpha\}$  converges to  $L^\alpha$ , for  $0 < \alpha \leq 1$ .

**11.2.13 Definition. Lipschitz Continuity** Let  $I$  denote a real interval. The function  $f : I \rightarrow \mathbf{R}$  is called *Lipschitz continuous* if it is Hölder continuous with exponent 1. That is, if there is a constant  $\gamma$ , such that

$$|f(x) - f(y)| \leq \gamma|x - y|$$

for all  $x$  and  $y$  in  $I$ . The constant  $\gamma$  is called the *Lipschitz* (or *Hölder*) *constant* of  $f$ .

## 11.2.2 Contractions

We introduce a class of functions that plays an important role in many iterative methods.

**11.2.14 Definition. Contractions** Let  $I$  be an interval. The function  $f : I \rightarrow I$  is called a *contraction* if it is Lipschitz continuous with Lipschitz constant  $\gamma < 1$ . In other words,  $f$  is a contraction if there exists a constant  $\gamma$ ,  $\gamma < 1$  such that

$$|f(x) - f(y)| \leq \gamma|x - y|$$

holds for all  $x$  and  $y$  in  $I$ .

**11.2.15 Theorem. Contraction Principle** Let  $I$  be a closed interval. If  $f : I \rightarrow I$  is a contraction, then

- (1) the equation  $x = f(x)$  has a unique solution,
- (2) for any  $x_0$  in  $I$  the sequence  $\{x_k\}$  defined by  $x_{k+1} = f(x_k)$  ( $k = 0, 1, 2, \dots$ ) converges to this solution.

*Proof.* Let  $f : I \rightarrow I$  be a given contraction, with  $I$  a closed subset of  $\mathbf{R}$ . We prove that the equation  $x = f(x)$  cannot have more than one solution: Suppose that  $x_*$  and  $y_*$  are any two solutions of  $x = f(x)$ . Then, we have

$$|x_* - y_*| = |f(x_*) - f(y_*)| \leq \gamma|x_* - y_*|.$$

If  $x_* \neq y_*$  the above gives  $\gamma \geq 1$ , a contradiction (since  $\gamma < 1$ ). We conclude that  $x_* = y_*$ .

Next, we prove that  $\{x_n\}$  is convergent. We have

$$|x_{n+2} - x_{n+1}| = |f(x_{n+1}) - f(x_n)| \leq \gamma|x_{n+1} - x_n|,$$

showing that the sequence  $\{x_n\}$  is contractive and, therefore, convergent [Note 10, Theorem 6.3]. Since  $\{x_n\}$  is a sequence in  $I$  and  $I$  is closed, the limit belongs to  $I$  also.

A contraction is continuous. Let  $x_*$  denote the limit of  $\{x_k\}$ . Then, the continuity of  $f$  gives

$$\lim_{n \rightarrow \infty} x_n = x_* \Rightarrow \lim_{n \rightarrow \infty} f(x_n) = f(x_*).$$

Therefore, since  $x_{n+1} = f(x_n)$ ,

$$x_* = \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} f(x_n) = f(x_*),$$

giving  $x_* = f(x_*)$ . That is,  $x_*$  satisfies the equation  $x = f(x)$ . □

**11.2.16 Example.** Compute the number

$$c = \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$$

Define  $f : [0, +\infty) \rightarrow \mathbf{R} : x \mapsto \sqrt{2 + x}$ . Define the sequence  $\{x_n\}$  by  $x_0 = 0$  and  $x_{n+1} = f(x_n)$  for  $n = 0, 1, 2, \dots$ . We have

$$\begin{aligned} |f(x) - f(y)| &= \left| \left( \sqrt{2+x} - \sqrt{2+y} \right) \frac{\sqrt{2+x} + \sqrt{2+y}}{\sqrt{2+x} + \sqrt{2+y}} \right| \\ &= \frac{|2+x - 2-y|}{\sqrt{2+x} + \sqrt{2+y}} \\ &\leq \frac{|x-y|}{2\sqrt{2}}, \end{aligned}$$

showing that  $f$  is a contraction. Therefore, the equation  $x = f(x)$  has a unique solution  $c$ , which satisfies  $c = \sqrt{2+c}$ . This gives  $c = 2$ .

### 11.2.3 Exercises

1. In the definition of a contraction, replace the condition  $\gamma < 1$  with  $\gamma \leq 1$ . Does the theorem still hold? Explain.
2. Find a function  $f : \mathbf{R} \rightarrow \mathbf{R}$  such that  $|f(x) - f(y)| < |x - y|$  and such that the equation  $x = f(x)$  has no solutions.

### 11.2.4 Continuity-Preserving Operations

**11.2.17 Theorem.** Let  $S$  be a subset of  $\mathbf{R}$  and let  $c$  be a limit point of  $S$ . If  $f, g : S \rightarrow \mathbf{R}$  are continuous at  $c$ , then each of the functions  $\alpha + f$ ,  $\alpha f$ ,  $f + g$ ,  $fg$ ,  $f/g$ ,  $|f|$ ,  $-f$ , defined in 1.2.1, is also continuous at  $c$ .

*Proof.* Apply Theorem 1.16. □

**11.2.18 Theorem.** Let  $S$  and  $T$  be subsets of  $\mathbf{R}$ . If  $f : S \rightarrow T$  is continuous at  $c \in S$  and  $g : T \rightarrow \mathbf{R}$  is continuous at  $f(c)$ , then the composition  $g \circ f : S \rightarrow \mathbf{R}$  is continuous at  $c$ .

*Proof.* Suppose that  $f$  and  $g$  are as stated in the hypotheses of the theorem. Given  $\epsilon > 0$  there is a  $\delta > 0$  such that

$$(1) \quad (\forall y \in T)(|y - f(c)| < \delta \Rightarrow |g(y) - g(f(c))| < \epsilon).$$

For the given  $\delta$  there is some  $\eta > 0$  such that

$$(2) \quad (\forall x \in S)(|x - c| < \eta \Rightarrow |f(x) - f(c)| < \delta).$$

Now,  $f(x) \in T$  for all  $x \in S$ . Therefore, from (1) and (2) we obtain

$$(\forall x \in S)(|x - c| < \eta \Rightarrow |f(x) - f(c)| < \delta \Rightarrow |g(f(x)) - g(f(c))| < \epsilon),$$

which says that  $g \circ f$  is continuous at  $c$ . □

### 11.2.5 Continuity and Open Sets

**11.2.19 Proposition.** Let  $S$  be a subset of  $\mathbf{R}$  and let  $c$  denote any point in  $S$ . For any function  $f : S \rightarrow \mathbf{R}$  the following two statements are equivalent:

- (1) The function  $f$  is continuous at the point  $c$ .
- (2) For every neighborhood  $J$  of  $f(c)$  there exists a neighborhood  $I$  of  $c$  such that

$$f_*(S \cap I) \subset J.$$

Note: The function  $f_*$  denotes the forward image function of  $f$ .

*Proof.* **We prove that (1) implies (2)** Suppose that (1) holds. That is, suppose that for every  $\epsilon > 0$  there is a  $\delta > 0$  such that

$$(3) \quad x \in S \wedge |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon.$$

Every neighborhood  $J$  of  $f(c)$  is of the form

$$J = (f(c) - \epsilon, f(c) + \epsilon) = \{y \in \mathbf{R} \mid |y - f(c)| < \epsilon\}$$

for some  $\epsilon > 0$ . By the continuity of  $f$  at  $c$ , there is a  $\delta > 0$  such that (3) holds. Define  $I = (c - \delta, c + \delta) = \{x \in \mathbf{R} \mid |x - c| < \delta\}$ . Then, (3) says

$$x \in S \wedge x \in I \Rightarrow f(x) \in J,$$

which implies (2).

**We prove that (2) implies (1)** Suppose that (2) holds. Then, given a neighborhood  $J$  of  $f(c)$  there is a neighborhood  $I$  of  $c$  such that

$$f_*(S \cap I) \subset J.$$

Therefore,

$$x \in S \wedge x \in I \Rightarrow f(x) \in J.$$

Now,

$$J = (f(c) - \epsilon, f(c) + \epsilon)$$

for some  $\epsilon > 0$  and  $I = (c - \delta, c + \delta)$  for some  $\delta > 0$ . Since  $J$  is arbitrary, so is  $\epsilon > 0$ . Further, since

$$f(x) \in J \Leftrightarrow |f(x) - f(c)| < \epsilon$$

and

$$x \in I \Leftrightarrow |x - c| < \delta,$$

we have shown that

$$x \in S \wedge |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon,$$

which says that  $f$  is continuous at  $c$ . □

**11.2.20 Remarks.** If  $f : S \rightarrow \mathbf{R}$  is continuous on  $S$ , the forward image  $f_*(T)$  of an open subset  $T$  of  $S$  is not necessarily an open set. For instance, consider  $f : \mathbf{R} \rightarrow \mathbf{R} : x \mapsto 1$ . No open subset of  $\mathbf{R}$  is mapped to an open set, since  $f_*(T) = \{1\}$  for every  $T \subset \mathbf{R}$ . Is the inverse image  $f^*(T)$  of an open set  $T$  an open set? The answer is a qualified yes. To illustrate why, consider  $f : [0, +\infty) \rightarrow \mathbf{R} : x \mapsto \sqrt{x}$ . We have  $f^*((-1, 1)) = [0, 1)$ , which is not open. But, we can salvage the situation if we make the notion of open set a relative notion, as shown below.

**11.2.21 Definition.** Let  $T \subset S \subset \mathbf{R}$ . We say that  $T$  is *open relative to  $S$*  if for every  $x \in T$  there is some  $\epsilon > 0$  such that

$$(1) \quad y \in S \wedge |y - x| < \epsilon \Rightarrow y \in T.$$

**11.2.22 Remark.** Let  $T \subset S \subset \mathbf{R}$ . If we remove “ $y \in S$ ” from (1) above, “ $T$  is open relative to  $S$ ” becomes “ $T$  is open,” which also means “ $T$  is open relative to  $\mathbf{R}$ .” In other words, the only notion we need is the notion of a set being open relative to another set.

**11.2.23 Proposition.** Let  $T \subset S \subset \mathbf{R}$ . Then, the two statements below are equivalent.

- (1)  $T$  is open relative to  $S$ .
- (2) There is an open subset  $R$  of  $\mathbf{R}$  such that  $T = S \cap R$ .

*Proof.* We prove that (1) implies (2) Suppose that (1) holds. For each  $x \in T$  there is some  $\epsilon_x > 0$  such that

$$y \in S \wedge |y - x| < \epsilon_x \Rightarrow y \in T.$$

For each  $x \in T$ , define  $R_x = \{y \in \mathbf{R} \mid |y - x| < \epsilon_x\}$ . Define  $R = \cup_{x \in T} R_x$ . Since each  $R_x$  is open, so is  $R$ .

Since  $x \in T$  implies  $x \in R_x \subset R$ , we have  $T \subset R \cap T$ . Since  $R_x \subset T$  for every  $x \in T$ , we have  $R_x \cap S \subset T$  for every  $x \in T$ . This gives  $R \cap S = \cup_{x \in T} R_x \cap S \subset T$ . We have shown that  $T = R \cap S$ .

**We prove that (2) implies (1)** Suppose that (2) holds. Then,  $T = S \cap R$  for some open set  $R$ . Since  $R$  is open and  $T \subset R$ , for every  $x \in T$  there is some  $\epsilon > 0$  such that  $I_x = \{y \in \mathbf{R} \mid |y - x| < \epsilon\} \subset R$ . Therefore,

$$T = S \cap R \wedge I_x \subset R \Rightarrow S \cap I_x \subset T,$$

which is equivalent to (1). □

### 11.2.6 Continuity and Compact Sets

To be continued...